Non-Abelian Black Holes with Magnetic Dipole Hair

Burkhard Kleihaus
Department of Mathematical Physics, University College, Dublin,
Belfield, Dublin 4, Ireland
Jutta Kunz
Fachbereich Physik, Universität Oldenburg, Postfach 2503
D-26111 Oldenburg, Germany

August 5, 2000

Abstract

We construct static axially symmetric black holes in SU(2) Einstein-Yang-Mills-Higgs theory. Located inbetween a monopole-antimonopole pair, these black holes possess magnetic dipole hair. The difference of their mass and their horizon mass equals the mass of the regular monopole-antimonopole solution, as expected from the isolated horizon framework.

Preprint hep-th/0008034

1 Introduction

In Einstein-Maxwell (EM) theory, black holes are completely determined by their mass, their charge and their angular momentum, i.e. black holes have no hair [1]. Furthermore, Israel's theorem of EM theory states, that static black holes are spherically symmetric. The "no-hair" theorem holds no longer in theories with non-abelian fields. Such theories possess whole sequences of neutral static spherically symmetric black hole solutions [2]. Furthermore, there are static black hole solutions with only axial symmetry [3], as well as black hole solutions with only discrete symmetries [4], so Israel's theorem holds neither in the presence of non-abelian fields.

The hairy black hole solutions are asymptotically flat and possess a regular event horizon [2]. Taking the radius of the event horizon to zero, globally regular particle-like solutions emerge. Recently, in the isolated horizon framework an intriguing relation between the mass of hairy black hole solutions and the mass of the corresponding regular solutions was found in Einstein-Yang-Mills (EYM) theory [5, 6]. Also, a modified uniqueness conjecture was presented [6].

SU(2) Einstein-Yang-Mills-Higgs (EYMH) theory, in particular, allows for a rich variety of globally regular particle-like solutions, such as gravitating monopoles [7] and multimonopoles [8] as well as gravitating monopole-antimonopole pairs [9]. Associated with the gravitating monopole solutions are non-abelian black hole solutions with "magnetic monopole hair", existing for not too large values of the event horizon [7]. Likewise, the gravitating multimonopole solutions give rise to magnetically charged non-abelian black hole solutions with hair [4].

Here we show, that the regular gravitating monopole-antimonopole pair (MAP) solutions [9] are similarly associated with hairy black hole solutions. Immersing a black hole symmetrically between the monopole and antimonopole results in a static axially symmetric black hole solution carrying "magnetic dipole hair". We further show that the difference of the mass and the horizon mass of these hairy black holes equals the mass of the regular MAP solution, as expected from the isolated horizon formalism [5].

2 Ansatz

We consider SU(2) EYMH theory with action

$$S = \int \left(\frac{R}{16\pi G} + \frac{1}{2e}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{4}\mathrm{Tr}(D_\mu\Phi D^\mu\Phi) \right) \sqrt{-g}d^4x \ , \tag{1}$$

and with Newton's constant G, Yang-Mills coupling constant e, Higgs vacuum expectation value η, and vanishing Higgs self-coupling. Variation with respect to the metric and the matter fields leads to the Einstein equations and the field equations, respectively.

The static axially symmetric black hole solutions with “magnetic dipole hair” are obtained in isotropic coordinates with metric [3]

$$ds^2 = -fdt^2 + \frac{m}{f}\left(dr^2 + r^2 d\theta^2\right) + \frac{l}{f} r^2 \sin^2\theta d\varphi^2 \ , \tag{2}$$

where f, m and l are only functions of r and θ. The MAP ansatz reads for the purely magnetic gauge field ($A_0 = 0$) [10, 9]

$$A_\mu dx^\mu = \frac{1}{2e}\left\{\left(\frac{H_1}{r}dr + 2(1-H_2)d\theta\right)\tau_\varphi - 2\sin\theta\left(H_3\tau_r^{(2)} + (1-H_4)\tau_\theta^{(2)}\right)d\varphi\right\} \tag{3}$$

and for the Higgs field

$$\Phi = \left(\Phi_1\tau_r^{(2)} + \Phi_2\tau_\theta^{(2)}\right) \ , \tag{4}$$

with $su(2)$ matrices (composed of the standard Pauli matrices τ_i)

$$\begin{aligned} \tau_r^{(2)} &= \sin 2\theta\tau_\rho + \cos 2\theta\tau_3 \ , \quad \tau_\theta^{(2)} = \cos 2\theta\tau_\rho - \sin 2\theta\tau_3 \ , \\ \tau_\rho &= \cos\varphi\tau_1 + \sin\varphi\tau_2 \ , \quad \tau_\varphi = -\sin\varphi\tau_1 + \cos\varphi\tau_2 \ . \end{aligned} \tag{5}$$

The four gauge field functions H_i and the two Higgs field functions Φ_i depend only on r and θ. Consistent with this ansatz, the general set of equations of motion is reduced to a set of nine elliptic partial differential equations. With respect to the residual gauge degree of freedom [11, 3, 10] we choose the gauge condition $r\partial_r H_1 - 2\partial_\theta H_2 = 0$ [10, 9].

3 Boundary conditions

We consider static axially symmetric black hole solutions with “magnetic dipole hair”, which are asymptotically flat, and possess a finite mass. Their boundary conditions at infinity and along the ρ- and z-axis are the same as those of the regular MAP solutions [9], i.e. at infinity the conditions are

$$H_1 = H_2 = 0 \ , H_3 = \sin\theta \ , 1 - H_4 = \cos\theta \ , \Phi_1 = \eta \ , \Phi_2 = 0 \ , f = m = l = 1 \ , \tag{6}$$

on the z-axis the functions H_1, H_3, Φ_2 and the derivatives $\partial_\theta H_2, \partial_\theta H_4, \partial_\theta \Phi_1, \partial_\theta f, \partial_\theta m, \partial_\theta l$ have to vanish, while on the ρ-axis the functions $H_1, 1-H_4, \Phi_2$ and the derivatives $\partial_\theta H_2, \partial_\theta H_3, \partial_\theta \Phi_1, \partial_\theta f, \partial_\theta m, \partial_\theta l$ have to vanish.

In order to obtain black hole solutions with a regular event horizon at radius $r_{\rm H}$, we impose the boundary conditions [3]

$$f = m = l = 0 \ , \quad r\partial_r\Phi_1 + H_1\Phi_2 = 0 \ , \quad r\partial_r\Phi_2 - H_1\Phi_1 = 0 \ , \tag{7}$$

$$\partial_\theta H_1 + r\partial_r H_2 = 0 \ , \quad r\partial_r H_3 - H_1 H_4 = 0, \quad r\partial_r H_4 + H_1(H_3 + {\rm ctg}\theta) = 0 \ , \tag{8}$$

where the boundary conditions on the matter field functions result from the equations of motion. Since one of the gauge field equations gives a boundary condition which coincides with the gauge fixing condition, $r\partial_r H_1 - \partial_\theta H_2 = 0$, we need to impose a further condition at the horizon to completely fix the gauge [3]. We choose $\partial_\theta H_1 = 0$.

From the equations of motion it follows [3], that the Kretschmann scalar is finite at the horizon, and that the surface gravity κ [12, 4],

$$\kappa^2 = -(1/4) g^{tt} g^{ij} (\partial_i g_{tt})(\partial_j g_{tt}) \ , \tag{9}$$

is constant, as required by the zeroth law of black hole physics.

4 Results

Introducing the dimensionless coordinate $x = r\eta e$ and the Higgs field $\phi = \Phi/\eta$, the equations depend only on the coupling constant α, $\alpha^2 = 4\pi G\eta^2$. The mass M of the black hole solutions can be obtained directly from the total energy-momentum "tensor" $\tau^{\mu\nu}$ of matter and gravitation, $M = \int \tau^{00} d^3 r$ [13], or equivalently from $M = -\int \left(2T_0{}^0 - T_\mu{}^\mu\right)\sqrt{-g}drd\theta d\varphi$, yielding the dimensionless mass $\mu/\alpha^2 = \frac{e}{4\pi\eta}M$.

Let us briefly recall the globally regular gravitating MAP solutions. In the limit $\alpha \to 0$, the lower branch of gravitating MAP solutions emerges smoothly from the flat space solution [10] and ends at the critical value $\alpha_{\rm cr}^{\rm reg} = 0.670$, when gravity becomes too strong for regular MAP solutions to persist [9]. However, at $\alpha_{\rm cr}^{\rm reg}$ a second branch of MAP solutions appears, extending back to $\alpha = 0$. Having higher masses, these MAP solutions constitute the upper branch of solutions. On both branches, the modulus of the Higgs field of the gravitating MAP solutions possesses two zeros, $\pm z_0$, on the z-axis, which correspond to the location of the monopole and antimonopole, respectively.

Let us now turn to the corresponding black hole solutions with "magnetic dipole hair", where a regular event horizon is located inbetween the monopole-antimonopole pair. These solutions are obtained by solving the set of equations of motion numerically, subject to the above boundary conditions [14].

For a fixed value of α, $0 < \alpha < \alpha_{\rm cr}^{\rm reg}$, we obtain two branches of black hole solutions. Imposing a regular event horizon at a small radius $x_{\rm H}$, the lower branch of black hole solutions emerges from the regular lower branch MAP solution. Along this lower branch, with increasing $x_{\rm H}$ the mass increases. Indeed, when a maximal value of the horizon radius $x_{\rm H,max}(\alpha)$ is reached, the mass increases further with decreasing $x_{\rm H}$, reaching a maximal value at $x_{\rm H,cr}(\alpha) < x_{\rm H,max}(\alpha)$. Decreasing $x_{\rm H}$ from $x_{\rm H,cr}(\alpha)$, a second branch of solutions appears. Along this upper branch with decreasing $x_{\rm H}$ the mass decreases, reaching the regular upper branch MAP solution, when $x_{\rm H} \to 0$.

We now introduce the area parameter x_Δ [5, 6], defined via the dimensionless area of the black hole horizon A,

$$A = 2\pi x_{\rm H}^2 \int_0^\pi d\theta \sin\theta \, \frac{\sqrt{lm}}{f}\bigg|_{x_{\rm H}^2} = 4\pi x_\Delta^2 \ . \tag{10}$$

When considered as a function of the horizon radius $x_{\rm H}$, the area parameter x_Δ also attains its maximal value at the critical value $x_{\rm H,cr}$. Therefore the dimensionless mass μ, when considered as a function of the area parameter x_Δ, attains its maximum value at the maximum value of x_Δ. Here the two branches merge, forming a spike, as seen in Fig. 1 for $4\pi\alpha^2 = 0.725$.

The maximal value of the horizon radius $x_{\rm H,max}(\alpha)$ increases with decreasing α. The weaker gravity, the bigger black holes are possible. Extrapolating to the limit $\alpha \to 0$, we obtain the biggest possible horizon radius, $x_{\rm H} = 0.1208$.

Let us now consider these black hole solutions in the deformed isolated horizon framework [5, 6]. For EYM theory an intriguing relation between the ADM mass μ of a black hole with area parameter x_Δ and the mass $\mu_{\rm reg}$ of the corresponding globally regular solution was found [5],

$$\mu = \mu_\Delta + \mu_{\rm reg} \ , \tag{11}$$

where the horizon mass μ_Δ is defined via

$$\mu_\Delta = \int_0^{x_\Delta} \kappa(x'_\Delta) x'_\Delta dx'_\Delta \ , \tag{12}$$

with rescaled surface gravity $\kappa \to \kappa/e\eta$.

We now consider this relation for the EYMH black holes with “magnetic dipole hair”. The inverse of the surface gravity κ, and the integrand $\kappa(x_\Delta)x_\Delta$ of the integral for the horizon mass, are shown in Fig. 2 for $4\pi\alpha^2 = 0.725$ as functions of the area parameter x_Δ for both branches of black hole solutions.

Evaluating the rhs of relation (11) for the black hole solutions along both branches, we obtain excellent agreement with the corresponding ADM masses, as seen in Fig. 1. Thus our numerical results indicate, that relation (11) also holds for the EYMH black hole solutions with “magnetic dipole hair”. (For each branch the corresponding regular solution is the reference point for the integration.) In particular, the mass of the regular upper branch solution is related to the mass of the regular lower branch solution via the horizon mass integral, performed along both branches.

Another crucial quantity in this formalism is the magnetic non-abelian charge of the horizon $P_\Delta^{\rm YM}$ [5, 6]

$$P_\Delta^{\rm YM} = \frac{1}{4\pi} \oint \sqrt{\sum_i \left(F^i_{\theta\varphi}\right)^2} d\theta d\varphi \ , \tag{13}$$

where the integral is over the surface of the horizon. We show $P_\Delta^{\rm YM}$ in Fig. 3. Also shown is the magnetic abelian charge of the horizon P_Δ^M, obtained analogously from the electromagnetic 't Hooft tensor $\mathcal{F}_{\mu\nu}$ [15]. Note, that the magnetic charge inside the horizon, $P^M = \oint \mathcal{F}_{\theta\varphi} d\theta d\varphi$, vanishes. These charges are of interest also, since in [6] a modified uniqueness conjecture for black holes was suggested, claiming that black holes are uniquely specified by their horizon charges.

Further details to these black hole solutions as well as to possible excited black hole solutions will be given elsewhere.

5 Outlook

Having constructed black hole solutions with "magnetic dipole hair", where the black hole resides at the center between the poles, it is natural to wonder, whether one could also obtain static axially symmetric solutions with two black holes, each located at one of the poles, i.e. black diholes [16]. Such solutions would presumably form an unstable equilibrium configuration of two hairy black holes.

When considering a magnetic dipole configuration with two black holes in Einstein-Maxwell theory, one has either to invoke a string between the holes or an external magnetic field to give the necessary repulsion and avoid collapse [16]. For the non-abelian configuration, in contrast, we expect the gauge field to provide the necessary repulsion. Indeed, for the globally regular MAP solutions and the above constructed black hole solutions with "magnetic dipole hair", the gauge field provides sufficient repulsion for equilibrium.

For the single black hole with "magnetic dipole hair" our numerical results are in excellent agreement with relation (11), obtained in the isolated horizon framework. It appears interesting to consider extension of this relation to the conjectured black dihole configurations.

References

[1] W. Israel, Commun. Math. Phys. **8** (1968) 245;
D. C. Robinson, Phys. Rev. Lett. **34** (1975) 905;
P. Mazur, J. Phys. **A15** (1982) 3173;
M. Heusler, Black Hole Uniqueness Theorems, (Cambrigde University Press, 1996).

[2] see e.g. M. S. Volkov and D. V. Gal'tsov, Gravitating Non-Abelian Solitons and Black Holes with Yang-Mills Fields, Phys. Rept. **319** (1999) 1.

[3] B. Kleihaus and J. Kunz, Phys. Rev. Lett. **79**, 1595 (1997);
B. Kleihaus and J. Kunz, Phys. Rev. **D57**, 6138 (1998).

[4] S. A. Ridgway and E. J. Weinberg, Phys. Rev. **D52** (1995) 3440.

[5] A. Ashtekar, S. Fairhurst, and B. Krishnan, preprint gr-qc/0005083.

[6] A. Corichi, and D. Sudarsky, gr-qc/9912032;
A. Corichi, U. Nucamendi, and D. Sudarsky, gr-qc/0002078.

[7] K. Lee, V.P. Nair and E.J. Weinberg, Phys. Rev. **D45** (1992) 2751;
P. Breitenlohner, P. Forgacs and D. Maison, Nucl. Phys. **B383** (1992) 357;
P. Breitenlohner, P. Forgacs and D. Maison, Nucl. Phys. **B442** (1995) 126.

[8] B. Hartmann, B. Kleihaus, J. Kunz, in preparation.

[9] B. Kleihaus and J. Kunz, preprint hep-th/0006148.

[10] Bernhard Rüber, Thesis, University of Bonn 1985;
B. Kleihaus and J. Kunz, Phys. Rev. **D61** (2000) 025003.

[11] C. Rebbi and P. Rossi, Phys. Rev. **D22** (1980) 2010;
B. Kleihaus, J. Kunz and D. H. Tchrakian, Mod. Phys. Lett. **A13** (1998) 2523.

[12] R. M. Wald, General Relativity (University of Chicago Press, Chicago, 1984)

[13] S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972)

[14] B. Kleihaus and J. Kunz, in preparation.

[15] G. ‘t Hooft, Nucl. Phys. **B79** (1974) 276.

[16] R. Emparan, Phys. Rev. **D61** (2000) 104009.

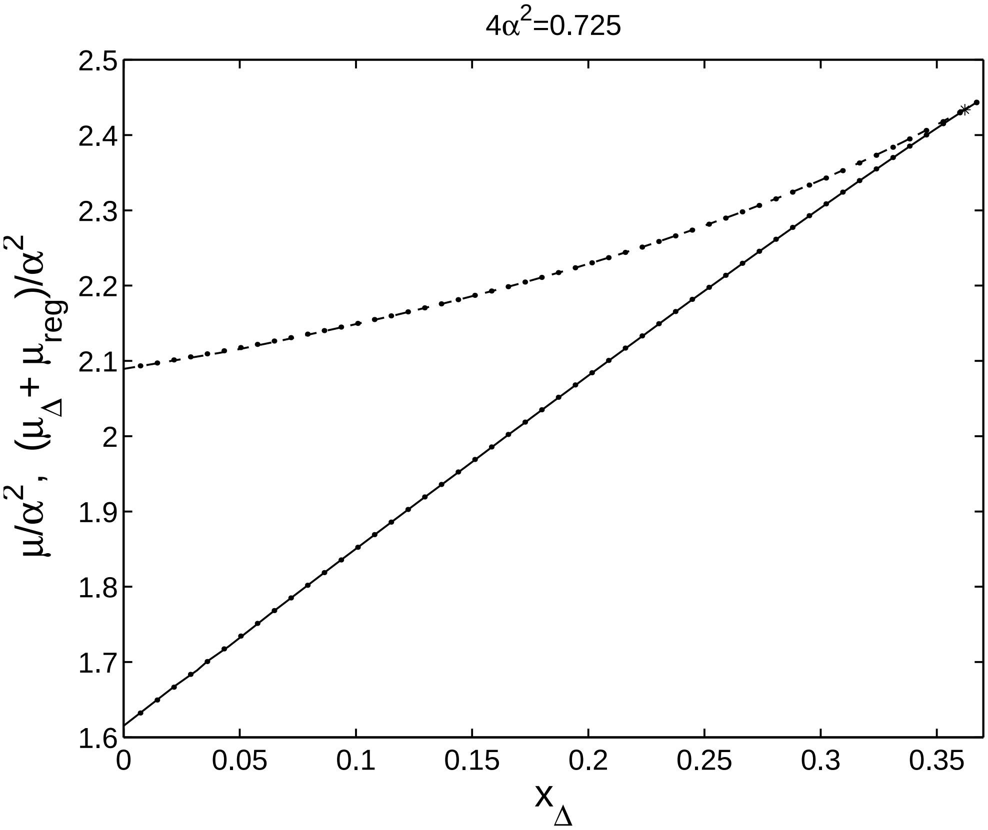

Figure 1: The ADM mass μ/α^2 along the lower branch (solid) and the upper branch (dashed) is shown for the EYMH black hole solutions with "magnetic dipole hair" as a function of the area parameter x_Δ for coupling constant $4\pi\alpha^2 = 0.725$. The asterisk $(*)$ indicates the solution with horizon radius $x_{\mathrm{H,max}}$. Also shown is the rhs of relation (11) $(\mu_\Delta + \mu_{\mathrm{reg}})/\alpha^2$ (dotted).

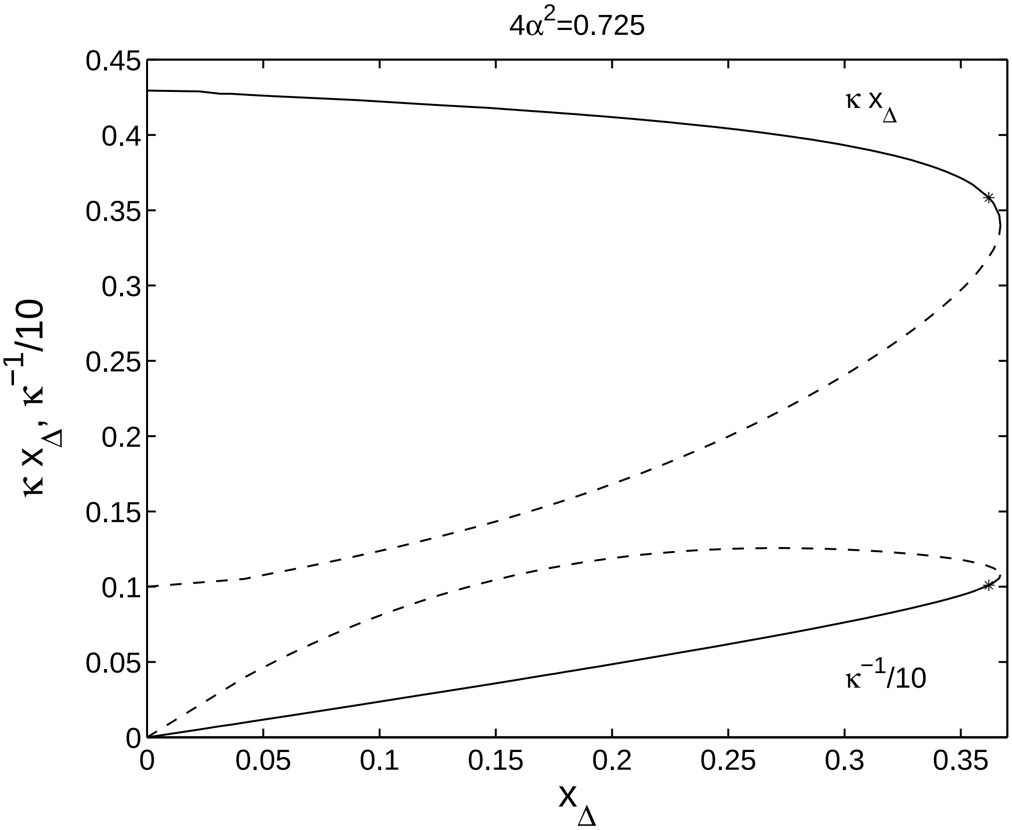

Figure 2: Same as Fig. 1 for the inverse (rescaled) surface gravity κ^{-1} and the integrand of the horizon mass κx_Δ. (The value of κx_Δ for $x_\Delta \to 0$ is extrapolated.)

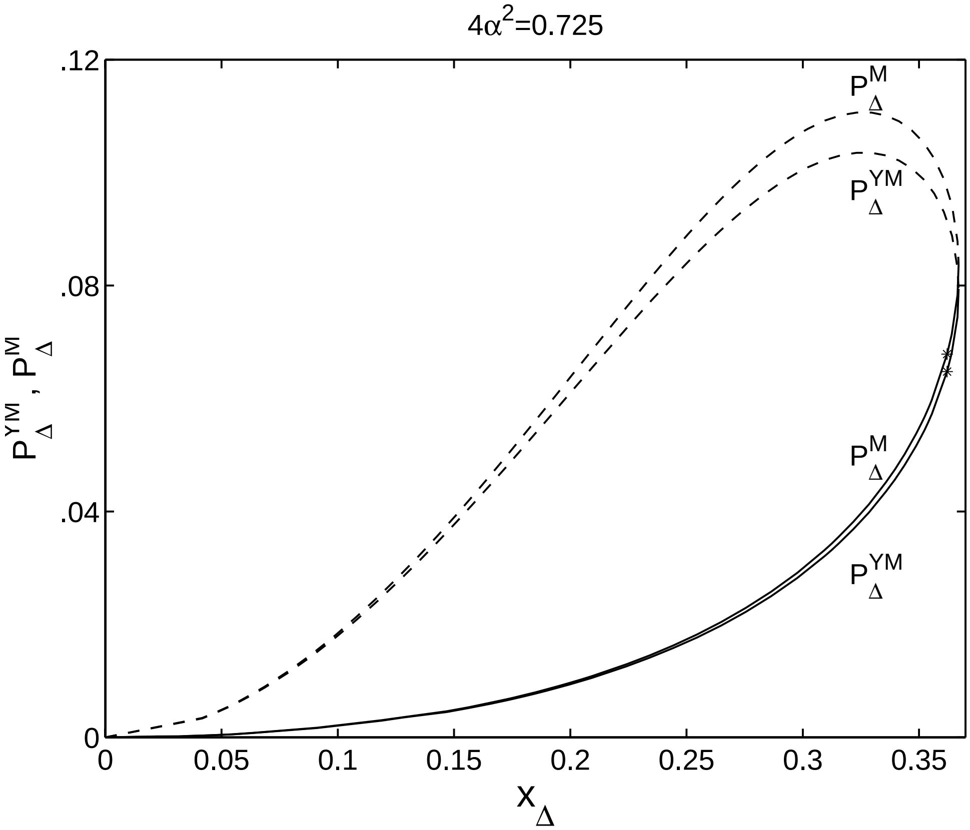

Figure 3: Same as Fig. 1 for the magnetic non-abelian charge of the horizon $P_\Delta^{\rm YM}$ and the magnetic abelian charge of the horizon $P_\Delta^{\rm M}$.